DETERMINING THE TEMPERATURE-DEPENDENCE OF THE THERMOPHYSICAL PROPERTIES OF SOLIDS BY THE METHOD OF SUCCESSIVE APPROXIMATIONS

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UDC 536.2.022

The nonlinear heat conduction problem is solved by the method of successive approximations and, in connection with it, the feasibility of determining the temperature characteristics of the thermophysical properties is also considered.

Present nonstationary methods of determining the thermophysical characteristics are based on solving the linear equation of heat conduction, which limits their applicability insofar as the operating temperature range must be sufficiently narrow: the more strongly the thermophysical properties depend on the temperature, the narrower must that range be. In order to extend the applicability of nonstationary methods, it is necessary to solve nonlinear equations of heat conduction.

Various approximate methods of solving nonlinear equations are known, but only a few of them are suitable for the determination of thermophysical characteristics. The method of a small parameter was used in [1-4], the method of integral substitutions was used in [5], and in [6, 7] the temperature field was sought in terms of power and functional series.

In this article we will use the solution to the nonlinear equation of heat conduction which has been obtained by the method of successive approximations.

The equation of heat conduction

$$C(t)\frac{\partial t}{\partial \tau} = \mathrm{div}\,(\lambda(t)\nabla t), \tag{1}$$

where

$$\lambda(t) = \lambda_0 + \sum_{k=1}^{l} \lambda_k (t - t_0)^k = \lambda_0 + \varphi(t), \tag{2}$$

$$C(t) = c_0 + \sum_{k=1}^{l} c_k (t - t_0)^k = c_0 + \psi(t), \tag{3}$$

can be restated as

$$c_0 \frac{\partial t}{\partial \tau} - \lambda_0 \Delta t = \mathrm{div}\,(\varphi(t)\nabla t) - \psi(t)\frac{\partial t}{\partial \tau}. \tag{4}$$

Let the initial condition

$$t|_{\tau=0} = t_0 \tag{5}$$

and the boundary conditions

$$\lambda(t)\frac{\partial t}{\partial x}\Big|_{x=0} = -q_0, \tag{6}$$

$$t|_{x=R} = t_0, \tag{7}$$

for (4) also be given.

Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 22, No. 4, pp. 718-724, April, 1972. Original article submitted July 31, 1970.

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Letting

$$\theta(x,\ \tau) = t(x,\ \tau) - t_0,\ \bar{x} = \frac{x}{R},\ \bar{\lambda}_k = \frac{\lambda_k}{c_0},\ \bar{c}_k = \frac{c_k}{c_0},$$

we may write Eq. (4) as follows:

$$\frac{\partial\theta}{\partial \mathrm{Fo}} - \frac{\partial^2\theta}{\partial\bar{x}^2} = \sum_{k=1}^{l}\frac{1}{k+1}\bar{\lambda}_k\frac{\partial^2\theta^{k+1}}{\partial\bar{x}^2} - \sum_{k=1}^{l}\frac{1}{k+1}\bar{c}_k\frac{\partial\theta^{k+1}}{\partial \mathrm{Fo}}. \tag{8}$$

Accordingly, the constraints will become

$$\theta|_{\mathrm{Fo}=0} = 0, \tag{9}$$

$$\left.\frac{\partial\theta}{\partial\bar{x}}\right|_{\bar{x}=0} = -\frac{q_0R}{\lambda_0} - \sum_{k=1}^{l}\frac{1}{k+1}\bar{\lambda}_k\left.\frac{\partial\theta^{k+1}}{\partial\bar{x}}\right|_{\bar{x}=0}, \tag{10}$$

$$\theta|_{\bar{x}=0} = 0. \tag{11}$$

Applying the integral transformation

$$T = \int_0^1 \theta\cos\mu_n\bar{x}\,d\bar{x} \tag{12}$$

to the system of Eqs. (8)-(11), we obtain the nonlinear first-order ordinary differential equation in T

$$\frac{dT}{d\mathrm{Fo}} - \frac{q_0R}{\lambda_0} + \mu_n^2T = -\mu_n^2\sum_{k=1}^{l}\frac{1}{k+1}\bar{\lambda}_k\int_0^1\theta^{k+1}\cos\mu_n\bar{x}\,d\bar{x} - \sum_{k=1}^{\infty}\frac{1}{k+1}\bar{c}_k\frac{d}{d\mathrm{Fo}}\int_0^1\theta^{k+1}\cos\mu_n\bar{x}\,d\bar{x} \tag{13}$$

with the zero boundary value:

$$T|_{\mathrm{Fo}=0} = 0, \tag{14}$$

where μ_n are the roots of the characteristic equation

$$\cos\mu_n = 0, \tag{15}$$

and the temperature expressed as

$$\theta^{k+1} = \left(2\sum_{k=1}^{\infty}T(\mathrm{Fo},\ n)\cos\mu_n\bar{x}\right)^{k+1} \tag{16}$$

is obtained from the inversion

$$\theta = 2\sum_{n=1}^{\infty}T(\mathrm{Fo},\ n)\cos\mu_n\bar{x}, \tag{17}$$

corresponding to the integral transformation (12). Equation (13) with the initial condition (14) will now be solved by successive approximations. As the zeroth approximation we take the solution to the equation

$$\frac{dT_0}{d\mathrm{Fo}} = \frac{q_0R}{\lambda_0} - \mu_n^2T_0 \tag{18}$$

with the initial condition

$$T_0|_{\mathrm{Fo}=0} = 0, \tag{19}$$

i.e.,

$$T_0 = \frac{q_0R}{\lambda_0}\cdot\frac{1}{\mu_n^2}(1 - \exp[-\mu_{n_0}^2\mathrm{Fo}]). \tag{20}$$

Inserting this into the right-hand side of Eq. (13), we obtain the linear differential equation

$$\frac{dT_1}{d\mathrm{Fo}} - \frac{q_0R}{\lambda_0} + \mu_{n_1}^2T_1 = -\mu_{n_1}^2\sum_{k=1}^{l}\frac{1}{k+1}\bar{\lambda}_k\int_0^1\theta_0^{k+1}\cos\mu_{n_1}\bar{x}\,d\bar{x}$$

$$-\sum_{k=1}^{l}\frac{1}{k+1}\bar{c}_k\frac{d}{d\mathrm{Fo}}\int_0^1\theta_0^{k+1}\cos\mu_{n_1}\bar{x}\,d\bar{x}, \tag{21}$$

whose solution with the initial condition

$$T_1|_{\text{Fo}=0} = 0 \tag{22}$$

represents the first approximation to Eq. (13).

Continuing this process, we obtain the solution to the i-th approximation in the form

$$T_i(\text{Fo}, n_i) = \frac{q_0 R}{\lambda_0} \cdot \frac{1}{\mu_{n_i}^2}(1 - \exp[-\mu_i^2 \text{Fo}])$$
$$- \exp[-\mu_{n_i}^2 \text{Fo}] \left[\mu_{n_i}^2 \sum_{k=1}^{l} \frac{1}{k+1} \bar{\lambda}_k \int_0^{\text{Fo}} \int_0^1 \theta_{i-1}^{p+1} \cos \mu_{n_i} \bar{x} \exp[\mu_{n_i}^2 \text{Fo}]\, d\bar{x} d\text{Fo}\right.$$
$$\left. + \sum_{k=1}^{l} \frac{1}{k+1} \bar{c}_k \int_0^{\text{Fo}} \int_0^1 \frac{d\theta^{k+1}}{d\text{Fo}} \cos \mu_{n_i} \exp[\mu_{n_i}^2 \text{Fo}]\, d\bar{x} d\text{Fo}, \right. \tag{23}$$

where μ_{n_i} are the roots of the characteristic equation

$$\cos \mu_{n_i} = 0. \tag{24}$$

From here the inversion formula

$$\theta_i(\bar{x}, \text{Fo}) = 2 \sum_{n_i=1}^{\infty} T_i(n_i, \text{Fo}) \cos \mu_{n_i} \bar{x}, \tag{25}$$

yields the final solution to the i-th approximation in the form

$$\theta_i(\bar{x}, \text{Fo}) = \theta_0(\bar{x}, \text{Fo}) + \bar{\lambda}_1 f_{1,i}(\bar{x}, \text{Fo}) + \cdots + \bar{\lambda}_k^{\varphi(i)} f_{k,i}(\bar{x}, \text{Fo}) + \cdots + \bar{c}_k^{\psi(i)} f_{m,i}(\bar{x}, \text{Fo}), \tag{26}$$

where $f_{m,i}(\bar{x}, \text{Fo})$, $\varphi(i)$, and $\psi(i)$ are functions determined from solution (25) with $m > 2k$. Solution (26) will be called a formal solution, since it is not feasible to prove the equality

$$\lim_{i \to \infty} \theta_i(\bar{x}, \text{Fo}) = \theta(\bar{x}, \text{Fo}) \tag{27}$$

on account of its unwieldiness. We will show how solution (26) can be used for determining the thermal conductivities and the specific heat. We assume that the temperature field at point $\bar{x} = 0$ is known from tests. Then setting up a system of m algebraic equations for point $\bar{x} = 0$ with $\text{Fo} = \text{Fo}_1, \text{Fo}_2, \ldots, \text{Fo}_m$,

$$\begin{aligned} \theta_i(0, \text{Fo}_1) &= \theta_0(0, \text{Fo}_1) + \bar{\lambda}_1 f_{1,i}(0, \text{Fo}_1) + \cdots + \bar{c}_k^{\psi(i)} f_{m,i}(0, \text{Fo}_1), \\ \theta_i(0, \text{Fo}_2) &= \theta_0(0, \text{Fo}_2) + \bar{\lambda}_1 f_{1,i}(0, \text{Fo}_2) + \cdots + \bar{c}_k^{\psi(i)} f_{m,i}(0, \text{Fo}_2), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \theta_i(0, \text{Fo}_m) &= \theta_0(0, \text{Fo}_m) + \bar{\lambda}_1 f_{1,i}(0, \text{Fo}_m) + \cdots + \bar{c}_k^{\psi(i)} f_{m,i}(0, \text{Fo}_m) \end{aligned} \tag{28}$$

and applying to them the Cramer rule, we find only the constants $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_k$; $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_k$, because it is not necessary to determine the constants $\bar{\lambda}_k^{\varphi(i)}$, $\bar{c}_k^{\psi(i)}$ and all their possible combinations for this particular case. Assuming λ_0 and c_0 to be given, from constants $\bar{\lambda}_k$, $\bar{c}_k$ we find constants λ_k, c_k within an accuracy defined by the condition that

$$\lim_{i \to \infty} |\theta_i(\bar{x}, \text{Fo}) - \theta_{i-1}(\bar{x}, \text{Fo})| = 0 \tag{29}$$

at any point $M(\bar{x}, \text{Fo})$. Inserting the value for λ_k and c_k into Eqs. (2) and (3), we find the sought temperature-dependence of thermal conductivity and specific heat.

We will now analyze the case where the temperature-dependence of thermal conductivity is assumed given and the temperature drop from point $\bar{x} = 0$ to point $\bar{x} = 1$ is negligible at any instant of time. Then, with the substitution

$$\theta = \frac{1}{\lambda_0} \int_{t_0}^{t} \lambda(t - t_0)\, d(t - t_0) \tag{30}$$

and considering that the thermal diffusivity varies according to

$$a(\theta) = a_0 - A\theta, \tag{31}$$

we may rewrite Eqs. (8)-(11) as

$$\frac{\partial\theta_i}{\partial \mathrm{Fo}} = \frac{\partial^2\theta_i}{\partial\bar{x}^2} - \bar{A}\theta_{i-1}\frac{\partial^2\theta_{i-1}}{\partial\bar{x}^2}, \tag{32}$$

$$\theta_i|_{\mathrm{Fo}=0} = 0, \tag{33}$$

$$\left.\frac{\partial\theta_i}{\partial x}\right|_{\bar{x}=0} = -\frac{q_0 R}{\lambda_0}, \tag{34}$$

$$\theta_i|_{\bar{x}=1} = 0, \tag{35}$$

where $\bar{A} = A/a_0$.

The solution to systems (32)-(35) at point $\bar{x} = 0$, $(\pi^2/4)\mathrm{Fo} = 1$ yields

$$\theta_1(0,\ \mathrm{Fo})\Big|_{\frac{\pi^2}{4}\mathrm{Fo}=1} = \frac{8\bar{q}}{\pi^2}\sum_{n_1=1}^{\infty}\frac{1}{(2n_1-1)^2}\left(1-\exp\left[-(2n_1-1)^2\right]\right)$$
$$-\bar{A}\left(\frac{8\bar{q}}{\pi^2}\right)^2\frac{1}{\pi}\sum_{n_1=1}^{\infty}\sum_{n_0=1}^{\infty}\sum_{n_0'=1}^{\infty}P(n_0,\ n_0',\ n_1)\,F_1(n_0,\ n_0',\ n_1); \tag{36}$$

$$\theta_2(0,\ \mathrm{Fo})\Big|_{\frac{\pi^2}{4}\mathrm{Fo}=1} = \frac{8\bar{q}}{\pi^2}\sum_{n_2=1}^{\infty}\frac{1}{(2n_2-1)^2}\left(1-\exp\left[-(2n_2-1)^2\right]\right)$$
$$-\bar{A}\left\{\left(\frac{8\bar{q}}{\pi^2}\right)^2\frac{1}{\pi}\sum_{n_2=1}^{\infty}\sum_{m_1=1}^{\infty}\sum_{n_1=1}^{\infty}P(n_1,\ m_1,\ n_2)\,F_2(n_1,\ m_1,\ n_2)\right.$$
$$-\bar{A}\left(\frac{8\bar{q}}{\pi^2}\right)^3\frac{1}{\pi^2}\sum_{n_2=1}^{\infty}\sum_{m_1=1}^{\infty}\sum_{n_1=1}^{\infty}\sum_{n_0=1}^{\infty}\sum_{n_0'=1}^{\infty}\frac{1}{(2n_0-1)^2}$$
$$\times P(n_0,\ n_0',\ n_1)\ P(n_1,\ m_1,\ n_2)\ F_3(n_0,\ n_0',\ n_1,\ m_1,\ n_2)$$
$$+\bar{A}\left(\frac{8\bar{q}}{\pi^2}\right)^3\frac{1}{\pi^2}\sum_{n_2=1}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_1=1}^{\infty}\sum_{m_0'=1}^{\infty}\sum_{m_0=1}^{\infty}\frac{(2m_1-1)^2}{(2m_0-1)^2(2n_1-1)^2}$$
$$\times P(m_0,\ m_0',\ m_1)\,P(n_1,\ m_1,\ n_2)\ F_4(m_0,\ m_0',\ m_1,\ n_1,\ n_2)$$
$$-\bar{A}^2\left(\frac{8\bar{q}}{\pi^2}\right)^4\frac{1}{\pi^3}\sum_{n_2=1}^{\infty}\sum_{m_1=1}^{\infty}\sum_{m_0=1}^{\infty}\sum_{m_0'=1}^{\infty}\sum_{n_1=1}^{\infty}\sum_{n_0=1}^{\infty}\sum_{n_0'=1}^{\infty}\frac{(2m_1-1)^2}{(2m_0-1)(2n_0-1)^2}$$
$$\times P(n_0,\ m_0',\ m_1)\ \ P(n_0,\ n_0',\ n_1)\ \ P(n_1,\ m_1,\ n_2)$$
$$\times F_5(n_0,\ n_0',\ n_1,\ m_0,\ m_0',\ m_1,\ n_2), \tag{37}$$

where

$$\bar{q} = \frac{q_0 R}{\lambda_0},\quad P(n_0,\ n_0',\ n_1)$$

$$= \frac{\sin\frac{\pi}{2}(2n_0+2n_0'-2n_1-1)}{2n_0+2n_0'-2n_1-1} + \frac{\sin\frac{\pi}{2}(2n_0'+2n_1-2n_0-1)}{2n_0'+2n_1-2n_0-1}$$
$$+\frac{\sin\frac{\pi}{2}(2n_1+2n_0-2n_0'-1)}{2n_1+2n_0-2n_0'-1} + \frac{\sin\frac{\pi}{2}(2n_0+2n_0'+2n_1-3)}{2n_0+2n_0'+2n_1-3}. \tag{38}$$

$P(n_1,\ m_1,\ n_2)$ and $P(m_0,\ m_0',\ m_1)$ are defined analogously

$$F_1(n_0,\ n_0',\ n_1) = \frac{1}{(2n_1-1)^2-(2n_0'-1)^2}\left(\exp\left[-(2n_0'-1)^2\right]-\exp\left[-(2n_1-1)^2\right]\right)$$
$$-\frac{1}{(2n_1-1)^2-(2n_0-1)^2-(2n_0'-1)^2}\left(\exp\left[-(2n_0-1)^2-(2n_0'-1)^2\right]-\exp\left[-(2n_1-1)^2\right]\right); \tag{39}$$

where the terms $F_2(n_1, m_1, n_2)$, $F_3(n_0, n_0', n_1, m_1, n_2)$, $F_4(m_0, m_0', m_1, n_1, n_2)$, $F_5(n_0, n_0', n_1, m_0, m_0', m_1, n_2)$ will not be written out explicitly because of their unwieldiness.

Solutions (36)-(37) indicate that the terms under the summation signs do not depend on test values. Therefore, they can be calculated beforehand for any given case. For instance, in our case ($\bar{x} = 0$, $(\pi^2/4)\mathrm{Fo} = 1$) these terms were calculated on a Minsk-22 computer:

$$\theta_1(0, \mathrm{Fo})\big|_{\frac{\pi^2}{4}\mathrm{Fo}=1} = \frac{8\bar{q}}{\pi^2}\Phi_1 - \bar{A}\left(\frac{8\bar{q}}{\pi^2}\right)^2 \frac{1}{\pi}\Phi_2, \tag{40}$$

$$\theta_2(0, \mathrm{Fo})\big|_{\frac{\pi^2}{4}\mathrm{Fo}=1} = \frac{8\bar{q}}{\pi^2}\Phi_1 - \bar{A}\left(\frac{8\bar{q}}{\pi^2}\right)^2 \frac{1}{\pi}\Phi_2 + \bar{A}^2\left(\frac{8\bar{q}}{\pi^2}\right)^3 \frac{1}{\pi^2}\Phi_3 - \bar{A}^2\left(\frac{8\bar{q}}{\pi^2}\right)^3 \frac{1}{\pi^2}\Phi_4 + \bar{A}^3\left(\frac{8\bar{q}}{\pi^2}\right)^4 \frac{1}{\pi^3}\Phi_5, \tag{41}$$

with $\Phi_1 = 0.8645563$, $\Phi_2 = 0.4184884$, $\Phi_3 = 0.2123646$, $\Phi_4 = 0.3629164$, $\Phi_5 = 0.1836947$. For most materials one may let $8\bar{q}/\pi^2 = 50$ and $\bar{A} = 0.002$ so that $\bar{A}^3(8\bar{q}/\pi^2)^4(1/\pi^3)\Phi_5 = 0.000296$. This shows that formula (41) is suitable for determining the thermal diffusivity with an accuracy entirely satisfactory for all practical purposes.

In an experiment with an infinitely large plate R = 11.22 mm thick and made of polymethylmethacrylate (density $\rho = 1282$ kg/m^3), $a_0 = 1.08335 \cdot 10^{-7}$ m^2/sec and $\lambda(t-t_0) = 0.181605 + B(t-t_0)$ W/m·°C, the value of $8\bar{q}/\pi^2$ was 59.654424. On the basis of these data, the first and the second approximation to A were found respectively from the linear and the quadratic equation in A of (41), i.e.,

$$\bar{A}_1 = 10^{-3}\cdot 2.208,\quad \bar{A}_2 = 10^{-3}\cdot 2.187,$$

and from here, taking into account expressions (26) and (27), we have the thermal diffusivity as a function of the temperature:

$$a(t-t_0) = a_0\left[1 - \bar{A}_2(t-t_0) + \frac{1}{2\lambda_0}B(t-t_0)^2\right], \tag{42}$$

where $B = 0.000277$ and $t_0 = 7$°C, which agrees within 5% with the data obtained at the VNIIM. Continuing in this manner, one can find coefficient $\bar{A}_i$ from the solution $\theta_i(0, \mathrm{Fo})|_{(\pi^2/4)\mathrm{Fo}=1}$ within an accuracy defined by condition (29).

This example shows that in our case it suffices to find $\bar{A}_2$. We note that the proposed method is applicable also where the thermophysical characteristics of solid cylinders or spheres are to be determined in either a one-dimensional or a two-dimensional analysis of the problem.

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